About Orbital Representation of Multidimensional Sets

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Abstract

In this work we state a new method of visualisation for point subsets on a multidimensional sphere for discrete geometry problems. Special features are invariancy of a set representation in a sphere movement and that visual image complexity only depends on the power of a set, but not dimension of a sphere.

Keywords Techniques for Multiple Dimensions, orbital histogram

1 Inroduction

It is desirable to have a visual representation of the complex structure objects in mathematical investigation. It's necessary for an extraction of the hidden properties of objects, for an identification of the isomorphic objects and for a control by the processes of their construction. Often it's insufficiently to apply common methods of visualization, so it's required to make special view, which intended for concrete visualisation problem. We have developed a view of the multidimensional sets, which arise in discrete geometry.

2 Subset of a multidimensional sphere as an object of visualisation

Further we will deal with finite sets on the $n-1$-dimensional sphere $S^{n-1} = \{x \in R^n : |x| = 1\}$. These sets are the object of discrete geometry research, for which the kissing number problem and the Tammes problem [1] are related. The *kissing number of space* R^n is τ_n – the greatest possible number of the points on a sphere S^{n-1} with minimum of an angle distance between them unless than 60 degrees. Today we know the kissing numbers only for 5 dimensions: 1, 2, 3, 8, 24 (2, 6, 12, 240, 196560 resp.)[1]. The Tammes problem is to find a set from S^{n-1} which consists of m points with the greatest possible *minimum angle* between the points of set. These problems have turned out extremely difficult, because we can't see directly the multidimensional structures, and so, to apply our geometric intuition. So we need tools for visual representation such sets. Common methods of representation (a projection to less dimensions; a representation of the points of set as the functions defined on set which power is n; the graph structures) are insufficient for our purposes. Using two first methods we can't understand, whether two sets "isomorphic" (because views depends on a coordinate system), *what* is disposition of the point related to another points of set; a complexity of an image grows with growth n. The methods using graph theory are independent on a coordinate system, but we lose obviousness of an image, and so – speed and simplicity of the visual interpretation. We have no such shortage in offering approach.

3 Orbital histogram

The basis of the method is our developed concept of *orbital histogram of point* w in respect of set $W \subset S^{n-1}$. This is the function $h^W_w(t)$, which value in $t \in [-1, 1]$ is a number of all points of W, that scalar product with w in R^n is equal t. If $w \in W$ we call h^W_w the *local histogram of* W. The *orbital histogram* $h^W(t)$ of set W is the sum of all local histograms of W. The *orbit of point* w *(of set* W*)* is a value of t, that $h^W_w(t) > 0$ ($h^W(t) > 0$).

Properties of the histograms: **(1)** h^W_w, h^W are invariant in a sphere movement, i.e. if set W is isomorphic to set U then $h^W = h^U$, $\{h^W_w\}_{w \in W} = \{h^U_u\}_{u \in U}$; **(2)** if W was obtained from a lattice, then all local histograms are equal; **(3)** the maximum orbit of W in $[0, 1)$ is the cosine of the minimum angle of W.

Let's associate the graphs of h^W and h^W_w ($w \in W$) to a set W. So we obtain the visual representation of W, which is independent of a coordinate system and which complexity doesn't grow with growth of n. We shall represent the applications of view in the examples.

4 Examples

We have developed the tool for visual research of the spherical sets. There we use two kinds of the representation: a projection and an orbital histogram. The fixed point which histogram represented marked on projection by special mark. Also the points from the orbit interesting us marked by special mark. In figures $h^W(1) = h^W_w(1) = 0$ for convenience.

1. In fig. 'A' the projection of set W_{24} from S^3, which contain 24 points, obtained from the lattice D_4 [1] is shown. Fig. 'B' demonstrate the graph of h_1 – the orbital histogram of point 1. In fig. 'A' point 1 and the points from its zero orbit are marked. The knowledge of points from an orbit let us orient in the projection of set because it provide us a metric structure of set which lose in the projection. We know that W_{24} was obtained from lattice. If we had no this information, we should suggested it, because we should verified that all local histograms are equal. Minimum angle is 60 degrees by property (3) and fig. 'B'.

2. In fig. 'C' the projection of the set W_{41} from S^4, consists of 41 points and with the minimum angle 59.0029024 degrees is shown [2], [3]. Because W_{41} has unequal local histograms (see fig. 'D', 'E'), it doesn't obtained from a lattice. It is clearly: the optimal set consist of 41 points can not be obtained from a lattice [1]. Some classes of points of W_{41} with an equal histograms were discovered. In fig. 'C' points of class $\{1, 10, 13\}$ are marked and in fig. 'D' its orbital histogram is shown. The histogram of the points of class $\{35, 41\}$ (see fig. 'E') is unexpected, because a small number of orbits is typical for points of a lattice set. Point 6 belong to another class $\{6, 7, 12\}$.

3. Fig. 'F' illustrate the regularity in the histogram of set W_{595} from S^9, which was built by random process. (Let's note that $500 \le \tau_{10} \le 595$ [1].)

5 Conclusion

Thus, we have two types of practical applications of orbital histograms. At first they give necessary conditions for a checking lattice of set and a (local) isomorphism of two sets. Secondly they let us analyse (to divide points into classes) and navigate (using orbits of point) in visual projection of set. Also they help in studying of the statistical regularities in random sets.

An image is invariant relative to movement of a sphere, the complexity of an image does not grow with growth of dimension. The method is suitable for analysis of the large sets. This facts and a histogram obviousness (in representing of a metric structure) makes our method the most useful while visual analysing of the spherical sets.

References

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