# Stackelberg Solutions for Nonzero-sum Differential Game in the Class of Nonanticipative Strategies

Yurii Averboukh, Artem Baklanov

Institute of Mathematics and Mechanics UrB RAS, Yekaterinburg, Russia
ayv@imm.uran.ru

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# Nonanticipative Strategies



*Nonanticipative Strategies was suggested for zero-sum differential games.*

C. Ryll-Nardzewski, S. Karlin, E. Roxin, R.J. Elliott, N.J. Kalton, P. Varayia and J. Lin, A.G. Chentsov et al.

# Outline

# Nonzero-sum differential game

$$\dot{x} = f(t, x, u) + g(t, x, v),$$
$$t \in [t_0, \vartheta_0], \quad x \in \mathbb{R}^n, \quad x(t_0) = x_0, \quad u \in P, \quad v \in Q.$$

$u$ is the control of the *Leader*;
$v$ is the control of the *Follower*.

*Purposes*

- The Leader wants to maximize $\sigma_L(x(\vartheta_0))$.
- The Follower wants to maximize $\sigma_F(x(\vartheta_0))$.

# Conditions

- The sets $P$ and $Q$ are compacts.
- The functions $f$, $\sigma_L$ and $\sigma_F$ are continuous;
- The function $f$ is locally Lipschitz continuous with respect to the phase variable;
- The function $f$ satisfies the sublinear growth condition with respect to $x$.
- The sets $\{f(t,x,u) : u \in P\}$, $\{g(t,x,v) : v \in Q\}$ are convex.

# Controls

$\mathcal{U} = \{u(\cdot) \text{ is measurable}\}$.
$\mathcal{V} = \{v(\cdot) \text{ is measurable}\}$.

*Nonanticipative Strategy of the Leader*
$\alpha : \mathcal{V} \to \mathcal{U}$ such that
the equality $v_1(t) = v_2(t)$ for $t \in [t_0, \tau]$ *implies*
the equality $\alpha[v_1](t) = \alpha[v_2](t)$ for $t \in [t_0, \tau]$.

# The motion

Denote by $x(\cdot, u(\cdot), v(\cdot))$ the solution of the initial value problem

$$\dot{x} = f(t, x, u(t)) + g(t, x, v(t)), \quad x(t_0) = x_0.$$

The strategy of the Leader $\alpha$ and the control of the Follower $v(\cdot)$ forms the motion $x[\cdot, \alpha, v(\cdot)]$:

$$\dot{x} = f(t, x, \alpha[v](t)) + g(t, x, v(t)), \quad x(t_0) = x_0.$$

# Stackelberg Solution

The Follower knows the nonanticipative strategy of the Leader $\alpha$.

*Set of the optimal controls of the Follower:*

$$V^b[\alpha] \triangleq \left\{ v_*(\cdot) : \sigma_F(x[\vartheta_0, \alpha, v_*(\cdot)]) = \max_{v(\cdot)\in\mathcal{V}} \sigma_F(x[\vartheta_0, \alpha, v(\cdot)]) \right\}.$$

# Stackelberg Solution

The strategy of the Leader $\alpha^*$ and the control of the Follower $v^*(\cdot)$ are the *Stackelberg Solution* if

- $v^*(\cdot) \in V^b[\alpha^*]$;
- $$\max_{\alpha} \max_{v(\cdot)\in V^b[\alpha]} \sigma_L(x(\vartheta_0, \alpha, v(\cdot)) = \sigma_L(x(\vartheta_0, \alpha^*, v^*(\cdot)]).$$

# Auxiliary zero-sum game

The Follower wants to maximize $\sigma_F(x(\vartheta_0))$, the purpose of the Leader is opposite. Denote the value of this game by $\omega_F : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$.

$$\omega_F(t_*, x_*) \triangleq \min_{\alpha} \max_{v(\cdot)} \sigma_F(x[\vartheta_0, t_*, x_*, \alpha, v(\cdot)]).$$

# Auxiliary set of controls

$$\mathcal{A} \triangleq \{(u(\cdot), v(\cdot)) : \omega_F(t, x(t)) \leq \omega_F(\vartheta_0, x(\vartheta_0)),\\ t \in [t_0, \vartheta_0], \quad x(\cdot) = x(\cdot, u(\cdot), v(\cdot))\}.$$

# Necessary Conditions

**Theorem**

If $(\alpha^*, v^*(\cdot))$ is the Stackelberg Solution then

- $(\alpha^*[v^*(\cdot)], v^*(\cdot)) \in \mathcal{A}$;
- $$\max\{\sigma_L(x(\vartheta_0, u(\cdot), v(\cdot))) : (u(\cdot), v(\cdot)) \in \mathcal{A}\} = \sigma_L(x[\vartheta_0, \alpha^*, v^*(\cdot)]).$$

# Sufficient Conditions

**Theorem**

Let $\hat{u}(\cdot) \in \mathcal{U}$ and $\hat{v}(\cdot) \in \mathcal{V}$ satisfy the following conditions

- $(\hat{u}(\cdot), \hat{v}(\cdot)) \in \mathcal{A}$;
- $$\max\{\sigma_L(x(\vartheta_0, u(\cdot), v(\cdot))) : (u(\cdot), v(\cdot)) \in \mathcal{A}\} \\ = \sigma_L(x(\vartheta_0, \hat{u}(\cdot), \hat{v}(\cdot))).$$

Then there exists the Stackelberg Solution $(\alpha^*, v^*(\cdot))$ such that
$$v^*(\cdot) = \hat{v}(\cdot), \quad \alpha^*[\hat{v}(\cdot)] = \hat{u}(\cdot).$$

# Existence

There exist at least one Stackelberg Solution $(\alpha^*, v^*(\cdot))$.

# Feedback Strategies



$u(t,x)$ is a feedback strategy of the Leader;
$v(t,x)$ is a feedback strategy of the Follower.

*The strategies $u(t,x)$ and $v(t,x)$ generate the motion $x(\cdot,u(\cdot,\cdot),v(\cdot,\cdot))$.*

# Stackelberg Solution in the class of Feedback Strategies



*The strategy of the Leader is known to the Follower.*

*Set of the optimal strategies of the Follower:*

$$V^b[u(\cdot,\cdot)] \triangleq \Big\{\hat{v}(\cdot,\cdot) : \sigma_F(x(\vartheta_0, u(\cdot,\cdot), \hat{v}(\cdot,\cdot))) \\ = \max_{v(\cdot)} \sigma_F(x(\vartheta_0, u(\cdot,\cdot), \hat{v}(\cdot,\cdot)))\Big\}.$$

# Stackelberg Solution in the class of Feedback Strategies



The pair of strategies $(u^*(\cdot,\cdot), v^*(\cdot,\cdot))$ is the *Stackelberg Solution* if

- $v^*(\cdot,\cdot) \in V^b[u^*(\cdot,\cdot)]$;
- $\sigma_L(x(\vartheta_0, u^*(\cdot,\cdot), v^*(\cdot,\cdot))) =$
$$\max_{u(\cdot,\cdot)} \max_{v(\cdot,\cdot)\in V^b[u(\cdot,\cdot)]} \sigma_L(x(\vartheta_0, u(\cdot,\cdot), v(\cdot,\cdot))).$$

# Characterization



Let $(u^*(\cdot,\cdot), v^*(\cdot,\cdot))$ be the Stackelberg Solution. Denote

$$x^*(\cdot) = x(\cdot, u^*(\cdot,\cdot), v^*(\cdot,\cdot));$$

$$u^*(t) = u^*(t, x^*(t)), \quad v^*(t) = v^*(t, x^*(t)).$$

*Properties*

- $(u^*(\cdot), v^*(\cdot)) \in \mathcal{A}$;
- $\max\{\sigma_L(x(\vartheta_0, u(\cdot), v(\cdot))) : (u(\cdot), v(\cdot)) \in \mathcal{A}\} =$
$$= \sigma_L(x(\vartheta_0), u^*(\cdot), v^*(\cdot)).$$

# Characterization



Let $u^*(\cdot)$ and $v^*(\cdot)$ satisfy the following properties

- $(u^*(\cdot), v^*(\cdot)) \in \mathcal{A}$;
- $\max\{\sigma_L(x(\vartheta_0, u(\cdot), v(\cdot))) : (u(\cdot), v(\cdot)) \in \mathcal{A}\}$ $= \sigma_L(x(\vartheta_0, u^*(\cdot), v^*(\cdot)))$.

Then there exists the Stackelberg Solution $(u^*(\cdot,\cdot), v^*(\cdot,\cdot))$ such that

$$u^*(t) = u^*(t, x^*(t)),\ v^*(t) = v^*(t, x^*(t))$$
$$\text{for } x^*(\cdot) = x(\cdot, u^*(\cdot,\cdot), v^*(\cdot,\cdot)).$$

# Example. The game

$$\begin{cases} \dot{x} & = & u \\ \dot{y} & = & v \end{cases}$$

$t \in [0,1]$, $u, v \in [-1,1]$,
$x(0) = x_0$, $y(0) = y_0$.

The Leader wants to maximize $\sigma_L(x,y) \triangleq x$.
The Follower wants to maximize $\sigma_F(x,y) \triangleq -|x-y|$.

# Auxiliary zero-sum differential game



$$\omega_F(t,x,y) = -|x-y|.$$

$\mathcal{A}$ is the set of controls $(u(\cdot), v(\cdot))$ satisfying the following condition.

- if $x_0 \geq y_0$ then $v(t) \geq u(t)$;
- if $y_0 \geq x_0$ then

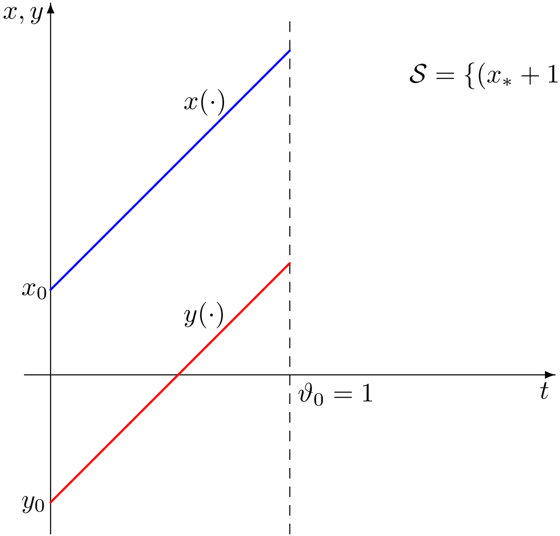
$$\begin{cases} u(t) \geq v(t), & t \leq \tau \\ u(t) = v(t), & t > \tau; \end{cases}$$

  here $\tau$ is defined by the equility

$$x_0 + \int_0^\tau u(t)dt = y_0 + \int_0^\tau v(t)dt.$$

# Case $x_0 > y_0$

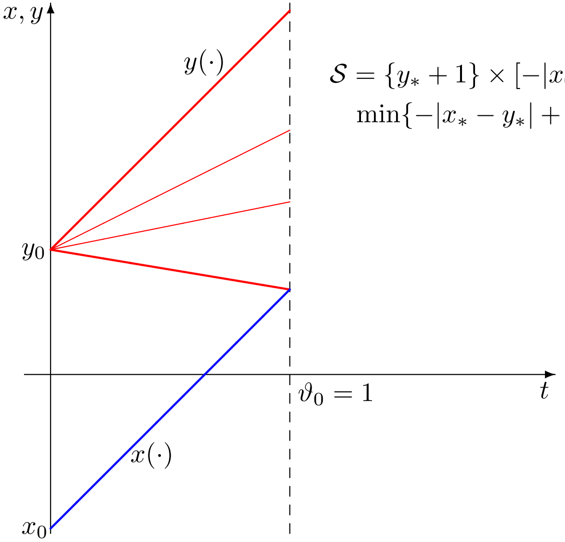





$$\mathcal{S} = \{(x_* + 1, -|x_* - y_*|)\}.$$

# Case $y_0 \geq x_0$






$$\mathcal{S} = \{y_* + 1\} \times [-|x_0 - y_0|, \min\{-|x_* - y_*| + 2, 0\}].$$