Universal Strategies with Guide for Nonzero-sum Differential Games

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Nonzero-sum differential game

$$\dot{x} = f(t, x, u) + g(t, x, v), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad u \in P, \quad v \in Q.$$

Here u and v are controls of the player I and the player II respectively.

- The player I wants to maximize $\sigma_1(x(T))$.
- The player II wants to maximize $\sigma_2(x(T))$.

Nash Equilibrium

Let $J_i(t_0, x_0, U, V)$ be the payoff of the i-th player at (t_0, x_0).

The pair (U^*, V^*) is a Nash Equilibrium at the position (t_0, x_0) if

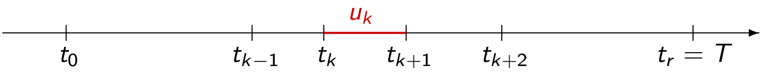
$$J_1(t_0, x_0, U, V^*) \leq J_1(t_0, x_0, U^*, V^*),$$
$$J_2(t_0, x_0, U^*, V) \leq J_2(t_0, x_0, U^*, V^*).$$

Approaches

- Punishment strategies:
  Existence of Nash equilibrium for a given initial position
  [Kononenko (1976), Tolwinski, Haurie and Leitmann (1986), Chistyakov (1981), Kleimenov (1993)].
- System of Hamilton-Jacobi equations:
  Smooth case [Friedman (1971)]
  Simple motions [Cardaliaguet, Plaskacz (2003)]
  Hyperbolicity case [Bressan, Wen Shen (2004a, 2004b)]

Control with Guide Strategies (Krasovkii, Subbotin)

Step-by-step-motion



- $u_k = u(t_k, x_k, w_k)$;
- w_k is called a guide, $w_k \in \mathbb{R}^m$;
- w_k is a function of t_k, t_{k-1}, x_{k-1} and w_{k-1}.

Outline

Conditions

- The sets P and Q are compacts.
- The functions f, g, σ_1 and σ_2 are continuous.
- The functions f and g are locally Lipschitz continuous with respect to the phase variable.
- The functions f and g satisfy the sublinear growth condition with respect to x.

Strategy of the player I

Strategy $U = (u, \psi_1, \chi_1)$*:*

- $u(t, x, w^1)$ forms the control;
- $\psi_1(t_+, t, x, w^1)$ is a transition function of the guide. It is equal to the state of the guide at t_+ under following condition: at the time instant t the state of the system is x, the state of the guide is w^1.
- $\chi_1(t_0, x_0)$ is a function initialing the guide.

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Strategy of the player II

Strategy $V = (v, \psi_2, \chi_2)$*:*

- $v(t, x, w^2)$ forms the control;
- $\psi_2(t_+, t, x, w^2)$ is a transition function of the guide. It is equal to the state of the guide at t_+ under the following condition: at time instant t the state of the system is x, the state of the guide is w^2.
- $\chi_2(t_0, x_0)$ is a function initialing the guide.

Motions generated by U

- U is a strategy of the player I;
- (t_0, x_0) is an initial position;
- $\Delta = \{t_0 < t_1 < \ldots < t_r = T\}$ is a partition of $[t_0, T]$;
- $v[\cdot]$ is a control of the player II.

Motion: $x^1[\cdot, t_0, x_0, U, \Delta, v[\cdot]]$.

- The state of the system at t_k denote by x_k, the state of the guide denote by w_k^1.
- The control of the player I on $[t_k, t_{k+1})$ is $u(t_k, x_k, w_k^1)$.
- $w_{k+1}^1 = \psi_1(t_{k+1}, t_k, x_k, w_k^1)$,
- $w_0^1 = \chi_1(t_0, x_0)$.

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Motions generated by V

- V is a strategy of the player II;
- (t_0, x_0) is an initial position;
- $\Delta = \{t_0 < t_1 < \ldots < t_r = T\}$ is a partition of $[t_0, T]$;
- $u[\cdot]$ is a control of the player I.

Motion: $x^2[\cdot, t_0, x_0, V, \Delta, u[\cdot]]$.

- The state of the system at t_k denote by x_k, the state of the guide denote by w_k^2.
- The control of the player II on $[t_k, t_{k+1})$ is $v(t_k, x_k, w_k^2)$.
- $w_{k+1}^2 = \psi_2(t_{k+1}, t_k, x_k, w_k^2)$,
- $w_0^2 = \chi_2(t_0, x_0)$.

Consistent motion

- U is a strategy of the player I;
- V is a strategy of the player II;
- (t_0, x_0) is an initial position
- $\Delta = \{t_0 < t_1 < \ldots < t_r = T\}$ is a partition of $[t_0, T]$.

Motion: $x^c[\cdot, t_0, x_0, U, V, \Delta]$.

Nash equilibrium

Upper Values

$$J_1^\natural(t_0, x_0, V) \triangleq \lim_{\delta\downarrow 0} \sup\{\sigma_1(x^2[T, t_0, x_0, V, \Delta, u[\cdot]]) : d(\Delta) \leq \delta\}$$

$$J_2^\natural(t_0, x_0, U) \triangleq \lim_{\delta\downarrow 0} \sup\{\sigma_2(x^1[T, t_0, x_0, U, \Delta, v[\cdot]]) : d(\Delta) \leq \delta\}$$

Lower Values

$$J_i^\flat(t_0, x_0, U, V) \triangleq \lim_{\delta\downarrow 0} \inf\{\sigma_1(x^c[T, t_0, x_0, U, V, \Delta]) : d(\Delta) \leq \delta\}.$$

Nash Equilibrium

The pair (U^*, V^*) *is a* ***Nash Equilibrium*** *at the position* (t_0, x_0) *if*

- $J_1^\natural(t_0, x_0, V^*) \leq J_1^\flat(t_0, x_0, U^*, V^*)$,
- $J_2^\natural(t_0, x_0, U^*) \leq J_2^\flat(t_0, x_0, U^*, V^*)$.

Notations

- $\mathcal{U} = \{u : [0, T] \to P \text{ measurable}\}$;
- $\mathcal{V} = \{v : [0, T] \to Q \text{ measurable}\}$.

Denote by $x(\cdot, t_*, x_*, u, v)$ the solution of initial value problem

$$\dot{x} = f(t, x, u, v), \quad x(t_*) = x_*$$

Sets of Motions

- $\mathrm{Sol}(t_*, x_*) \triangleq \mathrm{cl}\{x(\cdot, t_*, x_*, u, v) : u \in \mathcal{U}, v \in \mathcal{V}\}$;
- $\mathrm{Sol}^1(t_*, x_*; v) \triangleq \mathrm{cl}\{x(\cdot, t_*, x_*, u, v) : u \in \mathcal{U}\},\ v \in Q$;
- $\mathrm{Sol}^2(t_*, x_*; u) \triangleq \mathrm{cl}\{x(\cdot, t_*, x_*, u, v) : v \in \mathcal{V}\},\ u \in P$.

Theorem

Let the continuous functions $c_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2$ satisfy the following conditions:

(F1) $c_i(T, x) = \sigma_i(x)$, $i = 1, 2$;

(F2) for all $(t, z) \in [0, T] \times \mathbb{R}^n$, $u \in P$, $t_+ \in [t, T]$ there exists a motion $y(\cdot) \in \mathrm{Sol}^2(t, z; u)$ such that $c_1(t_+, y(t_+)) \leq c_1(t, z)$;

(F3) for all $(t, z) \in [0, T] \times \mathbb{R}^n$, $v \in Q$, $t_+ \in [t, T]$ there exists a motion $y(\cdot) \in \mathrm{Sol}^1(t, z; v)$ such that $c_2(t_+, y(t_+)) \leq c_2(t, z)$;

(F4) for all $(t, z) \in [0, T] \times \mathbb{R}^n$, $t_+ \in [t, T]$ there exists a motion $y(\cdot) \in \mathrm{Sol}(t, z)$ such that $c_i(t_+, y(t_+)) = c_i(t, z)$, $i = 1, 2$.

Theorem

Then for any compact set $G \subset [0, T] \times \mathbb{R}^n$ there exist CGS U^* and V^* providing Nash equilibria at all positions $(t_0, x_0) \in G$.

The i-th player's payoff at the position (t_0, x_0) *is*

$$J_i^\flat(t_0, x_0, U^*, V^*) = c_i(t_0, x_0).$$

System of Hamilton Jacobi Equations

Let $$\mathcal{H}_i(t, x, p_1, p_2) = \langle p_i, f(t, x, \hat{u}) + g(t, x, \hat{v})\rangle;$$

$$\max_u \langle p_1, f(t, x, u)\rangle = \langle p_1, f(t, x, \hat{u})\rangle,$$

$$\max_v \langle p_2, g(t, x, v)\rangle = \langle p_2, g(t, x, \hat{v})\rangle.$$

If (φ_1, φ_2) is the solution of the system

$$\frac{\partial \varphi_i}{\partial t} + \mathcal{H}_i(t, x, \nabla\varphi_1, \nabla\varphi_2) = 0, \quad i = 1, 2,$$

then φ_1, φ_2 satisfy conditions (F1)–(F4).

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Components of the Guide

- previous instant of correction: $\tau \in [0, T]$;
- residual: $d \in [0, +\infty)$;
- punishment component: $w^{(p)} \in \mathbb{R}^n$;
- consistent component: $w^{(c)} \in \mathbb{R}^n$.

Selected Motions

For any $(t, x) \in [0, T] \times \mathbb{R}^n$, $t_+ \in [t, T]$ choose $y^c(\cdot; t_+, t, x) \in \mathrm{Sol}(t, x)$ satisfying the conditions $c_i(t_+, y^c(t_+; t_+, t, x)) = c_i(t, x)$, $i = 1, 2$.

Notation

$$E = \mathrm{cl}\{x(t, t_0, x_0, u, v) : t \in [0, T], (t_0, x_0) \in G, u \in \mathcal{U}, v \in \mathcal{V}\}.$$

- L is a Lipschitz constant of $f + g$ on $[0, T] \times E \times P \times Q$ with respect to x.
- $K \triangleq \max\{\|f(t, x, u) + g(t, x, v)\| : x \in E, u \in P, v \in Q\};$
- $\varphi^*(\delta)$ is continuity modulus of $f + g$ with respect to t;
- $\beta \triangleq 2L;$
- $\varphi(\delta) = 4\mathrm{diam}(E)\varphi^*(\delta) + 4K^2\delta$

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Design of the Strategies

Let $w^i = (d^i, \tau^i, w^{i,(p)}, w^{i,(c)})$ be a state of the i-th player's guide at time instant t.

$$z^i \triangleq \begin{cases} w^{i,(c)}, & \|w^{i,(c)} - x\|^2 \leq \\ & \quad d^i(1+\beta(t-\tau^i)) + \varphi(t-\tau^i)(t-\tau^i), \\ w^{i,(p)}, & \text{otherwise.} \end{cases}$$

Design of the Strategies

- $u_* \in \operatorname{Argmax}\{\langle z^1 - x, f(t,x,u)\rangle : u \in P\}$;
- $v^* \in \operatorname{Argmin}\{\langle z^1 - x, g(t,x,v)\rangle : u \in Q\}$;
- $v_* \in \operatorname{Argmax}\{\langle z^2 - x, g(t,x,v)\rangle : u \in Q\}$;
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- $u(t,x,w^1) \triangleq u_*,\ v(t,x,w^2) \triangleq v_*$;
- $\psi_i(t_+,t,x,w^i) \triangleq w^i_+ = (d^i_+, \tau^i_+, w^{i,(p)}_+, w^{i,(c)}_+)$;
- $d^i_+ \triangleq \|z^i - x\|^2,\ \tau^i_+ \triangleq t$;
- $w^{i,(p)}_+ \triangleq y^i(t_+)$;
- $w^{1,(c)}_+ \triangleq y^{(c)}(t_+; t_+; t, z^1)$.

$y^1(\cdot)$ satisfies the condition (F3) with $z = z^1$, $v = v^*$,
$y^2(\cdot)$ satisfies the condition (F2) with $z = z^2$, $u = u^*$.

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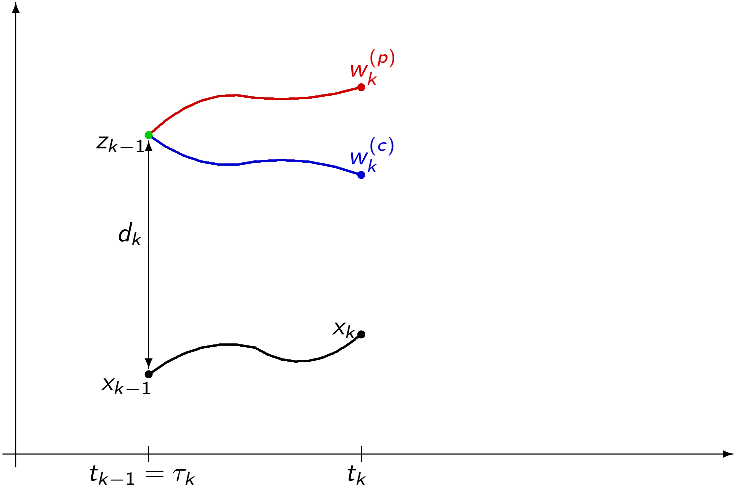
Design of the Strategies

$$\chi_i(t_0, x_0) \triangleq w_0^i = (d_0^i, \tau_0^i, w_0^{i,(p)}, w_0^{i,c}).$$

- $d_0^i = 0$;
- $\tau_0^i = t_0$
- $w_0^{i,(p)} = w_0^{i,c} = x_0$

Design of Strategies



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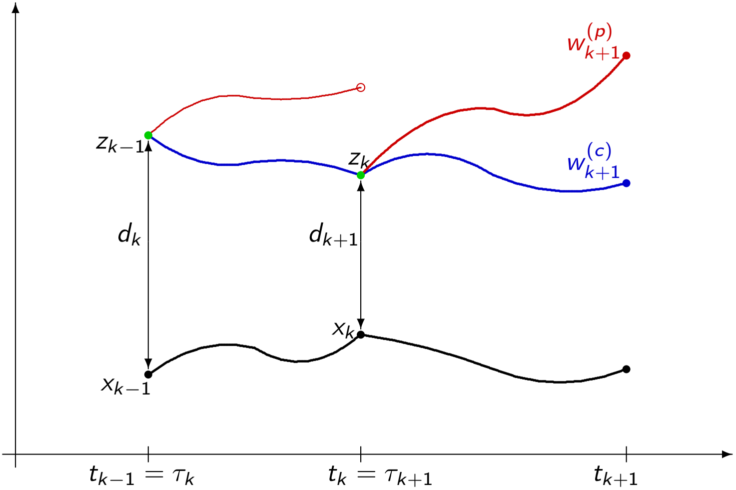

Property

Let $t_k, k = \overline{0, r}$ be a time instants of control correction,
x_k be a state of the system at t_k,
$w_k^i = (d_k^i, \tau_k^i, w_k^{i,(p)}, w_k^{i,(c)})$ be a state of the i-th player's guide.

Assume that the players use the strategies U^* and V^*.

Lemma

- $z_k^1 = w_k^{1,(c)} = w_k^{2,(c)} = z_k^2$;
- $\|x_k - w_k^{i,(c)}\|^2 \leq [(1 + (t_k - t_0))\varphi(d(\Delta))] \exp \beta(t_k - t_0)$;
- $c_i(t_k, w_k^{i,(c)}) = c_i(t_0, w_0^{i,(c)}) = c_i(t_0, x_0)$.

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- $c_i(t_k, w_k^{i,(c)}) = c_i(t_0, w_0^{i,(c)}) = c_i(t_0, x_0)$.

Equality

$$J_i^\flat(t_0, x_0, U^*, V^*) = \liminf_{d(\Delta)\downarrow 0} \sigma_i(x^c[T, t_0, x_0, U^*, V^*, \Delta]) = c_i(t_0, x_0)$$

Property

Assume that the player I uses the strategies U^*, when the player II deviates.

Lemma

- $\|x_k - z_k^1\|^2 \leq [(1 + (t_k - t_0))\varphi(d(\Delta))] \exp \beta(t_k - t_0);$
- $c_2(t_{k+1}, z_{k+1}^1) \leq c_2(t_k, z_k^1) \leq \ldots \leq c_2(t_0, z_0^1) = c_2(t_0, x_0).$

Main inequality:

$$J_2^\natural(t_0, x_0, U^*) \leq c_2(t_0, x_0) = J_2^\flat(t_0, x_0, U^*, V^*)$$
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Example

Dynamics:

$$\begin{cases} \dot{x}_1 = & v \\ \dot{x}_2 = & v + 2u \end{cases}$$

$t \in [0,1], u, v \in [-1,1].$

Goals:

- *I:* $x_1(1) \to \max$,
- *II:* $x_2(1) \to \max$.

System of Hamilton-Jacobi equations

System:

$$\begin{cases} \frac{\partial \varphi_1}{\partial t} - p_1 v^* + p_2(2u^* + v^*) & = 0 \\ \frac{\partial \varphi_2}{\partial t} - q_1 v^* + q_2(2u^* + v^*) & = 0 \end{cases}$$

Boundary condition: $\varphi_i(1, x_1, x_2) = x_i$.

Here $p_j = \partial\varphi_1/\partial x_j$, $q_j = \partial\varphi_2/\partial x_j$,
$u^* = \operatorname{sgn} p_2$, $v^* = \operatorname{sgn}(-q_1 + q_2)$.

Solutions:
Any pair $(\varphi_1, \varphi_2^\alpha)$ with $\varphi_1(t, x_1, x_2) = x_1 - (1 - t)$,
$\varphi_2^\alpha(t, x_1, x_2) = x_2 + (1 + 2\alpha)(1 - t)$
for $\alpha \in [-1, 1]$.

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Pair of functions

The pair (c_1, c_2) with $c_1 = x_1 + (1 - t)$,
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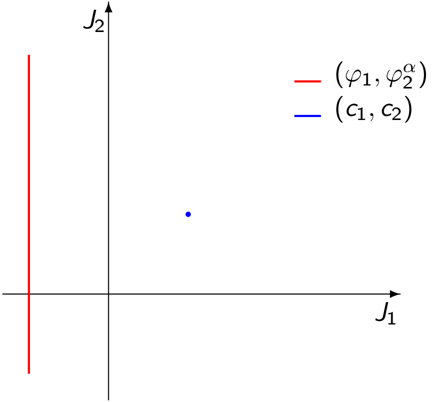
$c_2 = x_2 + (1 - t)$
satisfies conditions (F1)–(F4).



Example

Dynamics:

$$\dot{x} = u, \quad t \in [0,1], x \in \mathbb{R}, u \in [-1,1].$$

Goals:

- *I:* $|x(1)| \to \max$,
- *II:* $x(1) \to \max$.

The functions c_1 c_2

The value function of the player I

$$c_1 = |x| + (1 - t).$$

Optimal control of the player I

$$u^* \equiv \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

The value function of the player II

$$c_2(t,x) = \begin{cases} x+(1-t), & x > 0 \\ x-(1-t), & x < 0 \end{cases}$$

The functions c_1 c_2

The value function of the player I

$$c_1 = |x| + (1 - t).$$

Optimal control of the player I

$$u^* \equiv \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

The value function of the player II

$$c_2(t,x) = \begin{cases} x + (1 - t), & x > 0 \\ x - (1 - t), & x < 0 \end{cases}$$

Theorem

Let the upper semicontinuous multivalued map $S : [0, T] \times \mathbb{R}^n \multimap \mathbb{R}^2$ with nonempty values satisfy the following conditions

(S1) $S(T, x) = \{(\sigma_1(x), \sigma_2(x))\}$, $x \in \mathbb{R}^n$;

(S2) for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $(J_1, J_2) \in S(t, x)$, $u \in P$ and $t_+ \in [t, T]$ there exist the motion $y^2(\cdot) \in \mathrm{Sol}^2(t, x; u)$ and the pair of numbers $(J_1', J_2') \in S(t_+, y^2(t_+))$ such that $J_1 \geq J_1'$;

(S3) for all $(t_*, x_*) \in [0, T] \times \mathbb{R}^n$, $(J_1, J_2) \in S(t, x)$, $v \in Q$ and $t_+ \in [t, T]$ there exist the motion $y^1(\cdot) \in \mathrm{Sol}^1(t, x; v)$ and the pair of numbers $(J_1'', J_2'') \in S(t_+, y^1(t_+))$ such that $J_2 \geq J_2''$;

(S4) for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $(J_1, J_2) \in S(t, x)$ and $t_+ \in [t, T]$ there exists the motion $y^c(\cdot) \in \mathrm{Sol}(t, x)$ such that $(J_1, J_2) \in S(t_+, y^c(t_+))$.

Theorem

Then for any compact set $G \subset [0, T] \times \mathbb{R}^n$ there exist CGS U^* and V^* providing Nash equilibria at all positions $(t_0, x_0) \in G$. Moreover,

$$(J_1^\flat(t_0, x_0, U^*, V^*), J_2^\flat(t_0, x_0, U^*, V^*)) \in S(t_0, x_0).$$

Components of the Guide

- previous time of correction: $\tau \in [0, T]$;
- residual: $d \in [0, +\infty)$;
- expected payoff: $(\xi_1, \xi_2) \in \mathbb{R}^2$;
- punishment component of the guide: $w^{(p)} \in \mathbb{R}^n$;
- consistent component of the guide: $w^{(c)} \in \mathbb{R}^n$.

Existence Theorem

Theorem

There exists a multivalued functions $S : [0, T] \times \mathbb{R}^n \multimap \mathbb{R}^2$ *satisfying conditions* (S1)–(S4).

Proof is based on time discretization.

Discrete Time Game

$$y_{k+1} = f_k(y_k, u_k) + g_k(y_k, v_k),$$
$$y_k, y_{k+1} \in \mathbb{R}^n, k \in \overline{0, N-1}, \quad u_k \in P, \quad v_k \in Q.$$

- *The Player I maximizes $\sigma_1(y_N)$.*
- *The Player II maximizes $\sigma_2(y_N)$.*

The functions f_k and g_k are continuous.

Discretization

Fix N. Time step: $\delta = T/N$.

$$f_k(x,u) \triangleq \delta f(t_k, x, u),$$
$$g_k(x,v) \triangleq \delta g(t_k, x, v).$$

Value function for Discrete Time Games

There exist the upper semicontinuous functions $Z_k^N = Z_k : \mathbb{R}^n \multimap \mathbb{R}^2$ with nonempty values satisfying the following conditions:

1. $Z_N(y) = \{\sigma_1(y), \sigma_2(y)\}$;
2. for all $k \in \overline{0, N-1}$, $y \in \mathbb{R}^n$, $u \in P$, $(\xi_1, \xi_2) \in Z_k(y)$ there exist $v \in Q$ and $(\xi_1', \xi_2') \in Z_{k+1}(y + f_k(y,u) + g_k(y,v))$ such that $\xi_1 \geq \xi_1'$;
3. for all $k \in \overline{0, N-1}$, $y \in \mathbb{R}^n$, $v \in Q$, $(\xi_1, \xi_2) \in Z_k(y)$ there exist $u \in P$ and $(\xi_1'', \xi_2'') \in Z_{k+1}(y + f_k(y,u) + g_k(y,v))$ such that $\xi_2 \geq \xi_2''$;
4. for all $k \in \overline{0, N-1}$, $y \in \mathbb{R}^n$, $(\xi_1, \xi_2) \in Z_k(y)$ there exist $u \in P$, $v \in Q$ such that $(\xi_1, \xi_2) \in Z_{k+1}(y + f_k(y,u) + g_k(y,u))$.

Piecewise Constant Function

The multivalued map $S_N : [0, T] \times \mathbb{R}^n \multimap \mathbb{R}^2$

- $S_N(t, x) \triangleq Z_k^N(x),\ t \in (t_{k-1}, t_k),\ k = \overline{1, N};$
- $S_N(t_k, x) \triangleq Z_k^N(x) \cup Z_{k+1}^N(x),\ k = \overline{0, N-1};$
- $S_N(t_N, x) \triangleq Z_N^N(x).$

Convergence

The graph of the set S_N:

$$\mathrm{Gr}_\nu S_N \triangleq \{(t, x, \xi_1, \xi_2) : t \in [0, T], \|x\| \leq \nu, (\xi_1, \xi_2) \in S(t, x)\}.$$

Lemma

- *There exists the sequence S_{N_j} such that $\mathrm{Gr}_{\nu+1} S_{N_j}$ converge to a set R_ν in Hausdorff metrics for each ν;*
- *for $\nu < r$, $\|x\| \leq \nu$ $(t, x, \xi_1, \xi_2) \in R_r$ iff $(t, x, \xi_1, \xi_2) \in R_\nu$;*
- *for $\|x\| \leq \nu$ there exist ξ_1, ξ_2 such that $(t, x, \xi_1, \xi_2) \in R_\nu$.*

The Limit Multivalued Function

For $\|x\| \leq \nu$ put

$$\tilde{S}(t,x) \triangleq \{(\xi_1,\xi_2) : (t,x,\xi_1,\xi_2) \in R_\nu\}.$$

The multivalued function $\tilde{S}$ satisfies the conditions (S1)–(S4).

Conclusion

- Universal Control with Guide Strategies
- Case of Continuous Value Function
- Fail of Existence of Continuous Value Function
- Case of Discontinuous Value Function
- Existence of Discontinuous Value Function