A structure of the set of differential games solutions

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13-th International Symposium on Dynamical Games and Application
Wroclaw, Poland, June 30 – July 3, 2008

Purpose and Problem

Purpose:

Determine the set of values of differential games.

Problem:

Let the function $\varphi(\cdot,\cdot) : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be given. Design finitely dimensional compacts P and Q, dynamic function $f : [t_0, \vartheta_0] \times \mathbb{R}^n \times P \times Q \to \mathbb{R}$ and payoff function $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ such that function $\varphi(\cdot,\cdot)$ is a value of differential game

$$\dot{x} = f(t, x, u, v),\ t \in [t_0, \vartheta_0],\ x \in \mathbb{R}, u \in P,\ v \in Q$$

with payoff functional $\sigma(x(\vartheta_0))$.

Conditions

Conditions on sets

P and Q are compacts in finitely dimensional space.

Conditions on f

F1. f is continuous;

F2. f is locally lipschitzian with respect to x;

F3. for all $t \in [t_0, \vartheta_0]$, $x \in \mathbb{R}^n$, $u \in P$, $v \in Q$

$$\|f(t, x, u, v)\| \leq \Lambda_f(1 + \|x\|)$$

Conditions on σ

S1. σ is continuous;

S2. for all $x \in \mathbb{R}^n$

$$|\sigma(x)| \leq \Lambda_\sigma(1 + \|x\|).$$

Hamiltonian of Differential Game

We consider differential games in the class of *quasi-strategies of the first player* (advantage of the first player).

$$H(t,x,s) \triangleq \max_{v\in Q}\min_{u\in P}\langle s, f(t,x,u,v)\rangle.$$

Properties of Hamiltionian

H1. (sublinear growth condition) for all $(t, x, s) \in [t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$

$$|H(t, x, s)| \leq \Lambda_f \|s\|(1 + \|x\|);$$

H2. for every bounded region $A \subset \mathbb{R}^n$ there exist function $\omega_A \in \Omega$ and constant L_A such that for all $(t', x', s'), (t'', x'', s'') \in [t_0, \vartheta_0] \times A \times \mathbb{R}^n$ the following inequality holds:

$$\|H(t', x', s') - H(t'', x'', s'')\| \leq \\ \leq \omega(t' - t'') + L_A \|x' - x''\| + \\ + \Lambda_f (1 + \inf\{\|x'\|, \|x''\|\}) \|s_1 - s_2\|;$$

H3. H is positively homogeneous with respect to the third variable: if $\alpha \geq 0$ then

$$H(t, x, \alpha s) = \alpha H(t, x, s).$$

Equation:

$$\frac{\partial\varphi(t,x)}{\partial t} + H\left(t, x, \frac{\partial\varphi(t,x)}{\partial x}\right) = 0;$$

Boundary condition:

$$\varphi(\vartheta_0, x) = \sigma(x).$$

Minimax Solution [A.I. Subbotin]

Function φ is a *minimax solution* if for all $(t,x) \in (t_0, \vartheta_0) \times \mathbb{R}^n$ the following inequalities hold:

$$a + H(t,x,s) \leq 0 \ \forall (a,s) \in D^-_{\mathrm{D}}\varphi(t,x);$$

$$a + H(t,x,s) \geq 0 \ \forall (a,s) \in D^+_{\mathrm{D}}\varphi(t,x);$$

Dini Subdifferential

Lower Dini Derivative

Let $\tau \in \mathbb{R}$, $g \in \mathbb{R}^n$.

$$d_{\mathrm{D}}^{-}\varphi(t,x;\tau,g) \triangleq \liminf_{\delta\to 0} \frac{\varphi(t+\delta\tau, x+\delta g) - \varphi(t,x)}{\delta}$$

Dini Subdifferential

$$D_{\mathrm{D}}^{-}\varphi(t,x) \triangleq \{(a,s) \subset \mathbb{R}\times\mathbb{R}^n : \forall(\tau,g)\in\mathbb{R}\times\mathbb{R}^n \\ a\tau + \langle s,g\rangle \leq d_{\mathrm{D}}^{-}\varphi(t,x;\tau,g)\}.$$

Dini Superdifferential

Upper Dini Derivative

Let $\tau \in \mathbb{R}$, $g \in \mathbb{R}^n$.

$$d_{\mathrm{D}}^{+}\varphi(t,x;\tau,g) \triangleq \limsup_{\delta\to 0} \frac{\varphi(t+\delta\tau, x+\delta g) - \varphi(t,x)}{\delta}$$

Dini Superdifferential

$$D_{\mathrm{D}}^{+}\varphi(t,x) \triangleq \{(a,s) \subset \mathbb{R}\times\mathbb{R}^n : \forall(\tau,g) \in \mathbb{R}\times\mathbb{R}^n \quad a\tau + \langle s,g\rangle \geq d_{\mathrm{D}}^{+}\varphi(t,x;\tau,g)\}.$$

Property

If $D^-_{\mathrm{D}}\varphi(t,x) \neq \varnothing$ and $D^+_{\mathrm{D}}\varphi(t,x) \neq \varnothing$ simultaneously, then $(t,x) \in J$ and

$$D^-_{\mathrm{D}}\varphi(t,x) = D^+_{\mathrm{D}}\varphi(t,x) = \{(\partial\varphi(t,x)/\partial t, \nabla\varphi(t,x))\}.$$

Here

- $(\partial\varphi(t,x)/\partial t, \nabla\varphi(t,x))$ is total derivative;
- J denotes the set of points x at which function φ is differentiable. By the Rademacher's theorem measure $[t_0, \vartheta_0] \times \mathbb{R}^n \setminus J$ is 0.

Clarke Derivatives

Lower Clarke derivative

$$d^-_{\mathrm{Cl}}\varphi(t,x;\tau,g) \triangleq \liminf_{x'\to x, t'\to t, \alpha\to 0} \frac{1}{\alpha}(\varphi(t'+\alpha\tau, x'+\alpha g) - \varphi(t',x')).$$

Upper Clarke derivative

$$d^+_{\mathrm{Cl}}\varphi(t,x;\tau,g) \triangleq \limsup_{x'\to x, t'\to t, \alpha\to 0} \frac{1}{\alpha}(\varphi(t'+\alpha\tau, x'+\alpha g) - \varphi(t',x')).$$

Clarke subdifferential

There exists convex compact $\partial_{\mathrm{Cl}}\varphi(t,x) \subset \mathbb{R}\times\mathbb{R}^n$ such that

$$d^-_{\mathrm{Cl}}\varphi(t,x;\tau,g) = \min_{(a,s)\in\partial_{\mathrm{Cl}}\varphi(t,x)} [a\tau + \langle s,g\rangle],$$

$$d^+_{\mathrm{Cl}}\varphi(t,x;\tau,g) = \max_{(a,s)\in\partial_{\mathrm{Cl}}\varphi(t,x)} [a\tau + \langle s,g\rangle].$$

Properties of Clarke subdifferentials

Inclusions

$$D^-_{\mathrm{D}}\varphi(t,x) \subset \partial_{\mathrm{Cl}}\varphi(t,x),\ \ D^+_{\mathrm{D}}\varphi(t,x) \subset \partial_{\mathrm{Cl}}\varphi(t,x).$$

Representation

$$\partial_{\mathrm{Cl}}\varphi(t,x) = \mathrm{co}\{(a,s) : \exists\{t_i,x_i\}_{i=1}^{\infty} \subset J : \\ a = \lim_{i\to\infty} \partial\varphi(t_i,x_i)/\partial t,\ \ s = \lim_{i\to\infty} \nabla\varphi(t_i,x_i)\}.$$

Main Idea

Let $\varphi : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be local lipschitzian function such that $\varphi(\vartheta_0, \cdot)$ satisfies sublinear growth condition.

Procedure

1. Design a set $\mathbb{E} \subset [t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$ and function $h : \mathbb{E} \to \mathbb{R}^n$ in accordance with the function φ.
2. If the set $\mathbb{E}$ and functions h and φ satisfy some conditions, function φ is a value of some differential game.
3. Extend h to the whole space $[t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$.
4. Design control spaces P, Q and a dynamical function f in accordance with the extension of h.

$\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2$;
$\mathbb{E}_i = \{(t, x, s) : (t, x) \in [t_0, \vartheta_0] \times \mathbb{R}^n, \quad s \in E_i(t, x)\} \quad i = 1, 2.$
Set-valued maps $E_1(t, x)$ and $E_2(t, x)$ are defined below.

Points of Differentiability

Let $(t,x)\in J$. Put

$$E_1(t,x)\triangleq\{\nabla\varphi(t,x)\};$$

$$h(t,x,\nabla\varphi(t,x))\triangleq-\frac{\partial\varphi(t,x)}{\partial t}.$$

Condition (E1)

For any position $(t_*,x_*)\notin J$ and any sequences $\{(t'_i,x'_i)\}_{i=1}^{\infty}$, $\{(t''_i,x''_i)\}_{i=1}^{\infty}\subset J$ such that $(t'_i,x'_i)\to(t_*,x_*)$, $i\to\infty$, $(t''_i,x''_i)\to(t_*,x_*)$, $i\to\infty$, the following implication holds:

$$\big(\lim_{i\to\infty}\nabla\varphi(t'_i,x'_i)=\lim_{i\to\infty}\nabla\varphi(t''_i,x''_i)\big)\Rightarrow$$
$$\big(\lim_{i\to\infty}h(t'_i,x'_i,\nabla\varphi(t'_i,x'_i))=\lim_{i\to\infty}h(t''_i,x''_i,\nabla\varphi(t''_i,x''_i))\big).$$

Points of nondifferentiability

Limit Directions

Let $(t,x)\notin J$. Put

$$E_1(t,x)\triangleq\{s\in\mathbb{R}^n:\exists\{(t_i,x_i)\}\subset J:\\ \lim_{i\to\infty}(t_i,x_i)=(t,x)\ \&\ \lim_{i\to\infty}\nabla\varphi(t_i,x_i)=s\}.$$

$E_1(t,x)$ is nonempty and bounded.

Hamiltonian in limit directions

$$h(t,x,s)\triangleq\lim_{i\to\infty}h(t_i,x_i,\nabla\varphi(t_i,x_i))\\ \forall\{(t_i,x_i)\}\subset J:\lim_{i\to\infty}(t_i,x_i)=(t,x)\ \&\ s=\lim_{i\to\infty}\nabla\varphi(t_i,x_i).$$

Property

$$\partial_{\mathrm{Cl}}\varphi(t,x)=\mathrm{co}\{(-h(t,x,s),s):s\in E_1(t,x)\}.$$

Designation

$$CJ^- \triangleq \{(t,x) \in (t_0,\vartheta_0) \times \mathbb{R}^n \setminus J : D^-_{\mathrm{D}}\varphi((t,x)) \neq \varnothing\};$$

$$CJ^+ \triangleq \{(t,x) \in (t_0,\vartheta_0) \times \mathbb{R}^n \setminus J : D^+_{\mathrm{D}}\varphi((t,x)) \neq \varnothing\}.$$

Property: $CJ^- \cap CJ^+ = \varnothing$.

If $(t,x) \in CJ^-$,

$E_2(t,x) \triangleq \{s \in \mathbb{R}^n : \exists a \in \mathbb{R} : (a,s) \in D^-_{\mathrm{D}}\varphi((t,x))\} \setminus E_1(t,x)$;

if $(t,x) \in CJ^+$,

$E_2(t,x) \triangleq \{s \in \mathbb{R}^n : \exists a \in \mathbb{R} : (a,s) \in D^+_{\mathrm{D}}\varphi((t,x))\} \setminus E_1(t,x)$;

if $(t,x) \in ([t_0,\vartheta_0] \times \mathbb{R}^n) \setminus (CJ^- \cup CJ^+)$

$E_2(t,x) \triangleq \varnothing$.

Designation

$$E(t,x) \triangleq E_1(t,x) \cup E_2(t,x).$$
$$E^\natural(t,x) \triangleq \{\|s\|^{-1}s : s \in E(t,x) \setminus \{0\}\}.$$

Subsets of $[t_0,\vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n$

$$\mathbb{E}_1 \triangleq \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E_1(t,x)\},$$
$$\mathbb{E}_2 \triangleq \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E_2(t,x)\},$$
$$\mathbb{E} \triangleq \mathbb{E}_1 \cup \mathbb{E}_2 = \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E(t,x)\}.$$
$$\mathbb{E}^\natural \triangleq \{(t,x,s) : (t,x) \in [t_0,\vartheta_0] \times \mathbb{R}^n, \quad s \in E^\natural(t,x)\}.$$

Main Result

Let $\varphi : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be local lipschitzian function such that $\varphi(\vartheta_0, \cdot)$ satisfies sublinear growth condition.

Theorem

Function φ is a value of some differential game with terminal payoff functional if and only if the function h defined on $\mathbb{E}_1$ is extendable on the set $\mathbb{E}_2$ such that conditions (E1)–(E4) hold. (Conditions (E2)–(E4) are defined below.)

Condition (E2)

If $(t, x) \in CJ^-$ then for any $s_1, \ldots s_{n+2} \in E_1(t, s)$ $\lambda_1, \ldots, \lambda_{n+2} \in [0, 1]$ such that $\sum \lambda_k = 1$, $(-\sum \lambda_k h(t, x, s_k), \sum \lambda_k s_k) \in D^- \varphi(t, x)$ the following inequality holds:

$$h\left(t, x, \sum_{k=1}^{n+2} \lambda_k s_k\right) \leq \sum_{k=1}^{n+2} \lambda_k h(t, x, s_k);$$

If $(t, x) \in CJ^+$ then for any $s_1, \ldots s_{n+2} \in E_1(t, s)$ $\lambda_1, \ldots, \lambda_{n+2} \in [0, 1]$ such that $\sum \lambda_k = 1$, $(-\sum \lambda_k h(t, x, s_k), \sum \lambda_k s_k) \in D^+ \varphi(t, x)$ the following inequality holds:

$$h\left(t, x, \sum_{k=1}^{n+2} \lambda_k s_k\right) \geq \sum_{k=1}^{n+2} \lambda_k h(t, x, s_k);$$

Condition (E3)

- if $0 \in E(t,x)$, then $h(t,x,0) = 0$;
- if $s_1 \in E(t,x)$ and $s_2 \in E(t,x)$ are codirectional (i.e. $\langle s_1, s_2\rangle = \|s_1\| \cdot \|s_2\|$), then

$$\|s_2\| h(t,x,s_1) = \|s_1\| h(t,x,s_2).$$

Function $h^\natural : \mathbb{E}^\natural \to \mathbb{R}$

$\forall (t,x) \in [t_0, \vartheta_0] \times \mathbb{R}^n \; \forall s \in E(t,x) \setminus \{0\}$

$$h^\natural(t,x,\|s\|^{-1}s) \triangleq \|s\|^{-1} h(t,x,s).$$

Condition (E4)

Sublinear growth condition

there exists $\Gamma > 0$ such that for any $(t, x, s) \in \mathbb{E}^\natural$ the following inequality is fulfilled

$$h^\natural(t, x, s) \leq \Gamma(1 + \|x\|).$$

Difference estimate

For every bounded region $A \subset \mathbb{R}^n$ there exist $L_A > 0$ and modulus of continuity ω_A such that for any $(t', x', s'), (t'', x'', s'') \in \mathbb{E}^\natural \cap [t_0, \vartheta_0] \times A \times \mathbb{R}^n$ the following inequality is fulfilled

$$\|h^\natural(t', x', s') - h^\natural(t'', x'', s'')\| \leq \omega_A(t' - t'') + \\ + L_A\|x' - x''\| + \Gamma(1 + \inf\{\|x'\|, \|x''\|\})\|s' - s''\|.$$

A method of extension

Let $\varphi : [t_0, \vartheta_0] \times \mathbb{R}^n \to \mathbb{R}$ be local lipschitzian function such that $\varphi(\vartheta_0, \cdot)$ satisfies sublinear growth condition.

Corollary

Suppose that h as function defined on $\mathbb{E}_1$ satisfies the condition $(E1)$. Suppose also that the extension of h on $\mathbb{E}_2$ given by the following rule is well defined: for all $(t, x) \in CJ^- \cup CJ^+$, $s \in E_2(t, x)$, $s_1, \ldots, s_{n+2} \in E_1(t, x)$, $\lambda_1, \ldots, \lambda_{n+2} \in [0, 1]$ such that $\sum \lambda_i = 1$ $\sum \lambda_i s_i = s$

$$h(t, x, s) \triangleq \sum_{i=1}^{n+2} \lambda_i h(t, x, s_i).$$

If function $h : \mathbb{E} \to \mathbb{R}$ satisfies the conditions (E3) and (E4), then φ is a value of some differential game with terminal payoff functional.

Positive Example

Let $n = 2$, $t_0 = 0$, $\vartheta_0 = 1$.

$$\varphi_1(t, x_1, x_2) = t + |x_1| - |x_2|.$$

For $x_1, x_2 \neq 0$	$h(t, x_1, x_2; \mathrm{sgn} x_1, \mathrm{sgn} x_2) = -1.$
For $x_1 = 0$, $x_2 \neq 0$	$h(t, 0, x_2; \pm 1, \mathrm{sgn} x_2) = -1.$
For $x_1 \neq 0$, $x_2 = 0$	$h(t, x_1, 0; \mathrm{sgn} x_1, \pm 1) = -1.$
For $x_1 = x_2 = 0$	$h(t, 0, 0; \pm 1, \pm 1) = -1.$

$$J = \{(t, x_1, x_2) : x_1 x_2 \neq 0\}.$$
$$CJ^- = \{(t, 0, x_2) : x_2 \neq 0\},$$
$$CJ^+ = \{(t, x_1, 0) : x_1 \neq 0\}.$$

The extension is designed with the help of Corollary.

Negative Example

Let $n = 2$, $t_0 = 0$, $\vartheta_0 = 1$.

$$\varphi_2(t, x_1, x_2) = t(|x_1| - |x_2|).$$

$$J = \{(t, x_1, x_2) : t \in (0,1), x_1 x_2 \neq 0\}.$$

For $(t, x) \in J$ $\quad E(t, x) = \{(t \cdot \mathrm{sgn} x_1, t \cdot \mathrm{sgn} x_2)\}.$

$$h(t, x_1, x_2; t \cdot \mathrm{sgn} x_1, t \cdot \mathrm{sgn} x_2) = |x_1| - |x_2|.$$

Sets

$$\mathbb{E}_0 \triangleq \{(t, x_1, x_2; t\mathrm{sgn} x_1, t\mathrm{sgn} x_2) : (t, x_1, x_2) \in J\}.$$

$$\mathbb{E}_0^\natural \triangleq \{(t, x_1, x_2; \mathrm{sgn} x_1/\sqrt{2}, \mathrm{sgn} x_2/\sqrt{2}) : (t, x_1, x_2) \in J\}.$$

Restriction of $h^\natural$ on $\mathbb{E}_0^\natural$. Let $(t, x_1, x_2) \in J$

$$h^\natural(t, x_1, x_2; \mathrm{sgn} x_1/\sqrt{2}, t\mathrm{sgn} x_2/\sqrt{2}) = \frac{|x_1| - |x_2|}{\sqrt{2}t}.$$

Scheme of Proof

Step 0

Define payoff functional by formula $\sigma(\cdot) \triangleq \varphi(\vartheta_0, \cdot)$

Step 1

- Extend function $h^\natural$ defined on $\mathbb{E}^\natural$ to the set $[t_0, \vartheta_0] \times \mathbb{R}^n \times S^{(n-1)}$. ($S^{(k)}$ is k-dimensional sphere). Denote this extension by h^*.
- Design the positively homogeneous function $H : [t_0, \vartheta_0] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is an extension of h^*.

Step 2

Design finitely dimensional compacts P, Q and function f in accordance with H.

Statements

Corollary

If $\varphi(\cdot,\cdot)$ is a value of differential game with advantage of the *first player*, then $\varphi(\cdot,\cdot)$ is a value of some differential game with advantage of the *second player*. The converse is also true.

Case $n = 1$

The set of values of all-possible differential games coincides with the set of values of differential game which satisfies Isaacs condition.

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